A note on boundary value problems for black hole evolution

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Abstract

In recent work of Allen et. al., heuristic and numerical arguments were put forth to suggest that boundary value problems for black hole evolution, where an appropriate Sommerfeld radiation condition is imposed, would probably fail to produce Price law tails. The interest in this issue lies in its possible implications for numerical relativity, where black hole evolution is typically studied in terms of such boundary formulations. In this note, it is shown rigorously that indeed, Price law tails do not arise in this case, i.e. that Sommerfeld (and more general) radiation conditions lead to decay faster than any polynomial power. Our setting is the collapse of a spherically symmetric self-gravitating scalar field. We allow an additional gravitationally coupled Maxwell field. The proof also applies to the easier problem of a spherically symmetric solution of the wave equation on a Schwarzschild or Reissner-Nordström background. The method relies on previous work of the authors.

One of the characteristic features of black holes forming in the context of gravitational collapse is the appearance of so-called *Price law tails.* These describe the decay of radiation on timelike curves in the exterior region of the black hole, as well as the decay of the radiation flux along the event horizon. In the context of radiation described by a scalar field, Price [16] put forth heuristic arguments indicating that this decay should be $\sim v^{-3}$, where v is an Eddington-Finkelstein-like advanced time coordinate. In [10], it was rigorously proven in the context of the coupled Einstein-scalar field equations under spherical symmetry (and more generally of the Einstein-Maxwell-scalar field equations) that

$$|\phi| + |\partial_v \phi| \leq C_{\epsilon,R} v^{-3+\epsilon}, \tag{1}$$

in the region $r \leq R$, for any $\epsilon > 0$ and sufficiently large R. The decay rates (1) also hold for spherically symmetric solutions of the wave equation on a fixed Schwarzschild or Reissner-Nordström background.

In the present paper, it is proven that if appropriate boundary conditions are imposed at $r = R$, for fixed R, then decay can be proven at rates faster than any given polynomial power. The conditions include standard Sommerfeld boundary conditions. This paper was motivated by [1], where the absense of

Price law tails was suggested on the basis of heuristic arguments and numerical calculations. In particular, this paper confirms the validity of the heuristics of [1].

In fact, the argument of this paper is a rather straightforward application of the method of [10]. In that paper, decay rates were achieved by an inductive argument, at each step of which, the rate was improved by an amount constrained by information coming from infinity. As we shall see below, with appropriate boundary conditions, there is no such constraint, and the decay rate can be improved at each step by a fixed finite polynomial power. Thus, the inductive argument yields that any polynomial power can be achieved.

1 The equations

The arguments of this paper apply to the Einstein-Maxwell-scalar field system under spherical symmetry. For an introduction to this system, the author can consult [8, 9]. The Einstein-scalar field system, whose rigorous study was initiated in [7], is of course a special case, where the Maxwell part vanishes. The results of this paper also apply in the case where the wave equation is decoupled from the Einstein equations. For more on the relationship between these two problems, see [10].

Recall that the quotient manifold $\mathcal{Q} = \mathcal{M}/SO(3)$ inherits a 1+1-dimensional Lorentzian metric, which in null coordinates takes the form $-\Omega^2 dudv$. The metric of $\mathcal{M}$ can then be written $-\Omega^2 dudv + r^2\gamma$, where γ is the standard metric on S^2, and r is the area radius function $r : \mathcal{Q} \to \mathbf{R}$ defined by $r(q) = \sqrt{Area(q)/4\pi}$. We recall the Hawking mass $m = \frac{r}{2}\left(1 + 4\Omega^{-2}\partial_u r \partial_v r\right)$, and the mass aspect function $\mu = \frac{2m}{r}$. Under spherical symmetry, the Maxwell equations decouple, contributing a $\frac{e^2}{r^4}\Omega^2 dudv + \frac{e^2}{2r^2}\gamma$ term to the energy momentum tensor, where e is a constant. Defining the "renormalized" Hawking mass $\varpi = m + \frac{e^2}{2r}$, the evolution of the metric and scalar field are then completely determined by the following sytem of equations:

$$\partial_u r = \nu, \tag{2}$$

$$\partial_v r = \lambda, \tag{3}$$

$$\partial_u \lambda = \lambda \left(\frac{2\nu}{1-\mu} \left(\frac{\varpi}{r^2} - \frac{e^2}{r^3} \right) \right), \tag{4}$$

$$\partial_v \nu = \nu \left(\frac{2\lambda}{1-\mu} \left(\frac{\varpi}{r^2} - \frac{e^2}{r^3} \right) \right), \tag{5}$$

$$\partial_u \varpi = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu, \tag{6}$$

$$\partial_v \varpi = \frac{1}{2}(1-\mu)\left(\frac{\theta}{\lambda}\right)^2 \lambda, \tag{7}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{8}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{9}$$

We also easily obtain from the above the equations

$$\partial_u \frac{\lambda}{1-\mu} = \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu \frac{\lambda}{1-\mu} \tag{10}$$

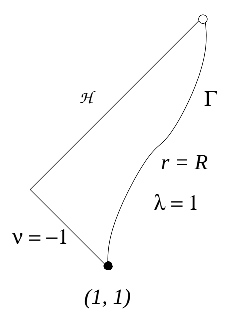
and

$$\partial_v \frac{\nu}{1-\mu} = \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 \lambda \frac{\nu}{1-\mu}. \tag{11}$$

2 The assumptions

We will assume the existence of a spacetime bounded by an ingoing null curve, a timelike curve, and an outgoing null curve, which we will call the *event horizon*.[1] Certain conditions will be imposed on the initial ingoing null curve and the timelike boundary. In this paper, we will not concern ourselves with the explicit *construction* of solutions satisfying the conditions described here, although this can be accomplished easily enough using the methods of [11, 10].

We consider a spacetime $\mathcal{Q}$ with Penrose diagram depicted below, where Γ is a timelike boundary:



Here, $\Gamma \cup \mathcal{H} \subset \mathcal{Q}$, but $\Gamma \cap \mathcal{H} = \emptyset$. The point of intersection of the initial ingoing ray and Γ will be $(1,1)$. The ingoing ray will then be the curve $v = 1$. The equations (2)–(9) are assumed to hold pointwise. On Γ, $\lambda = 1$, $r = R$ for some constant R, and m will be non-increasing as a function of v, such that $M_0 = 2\sup_\Gamma m < R$. Moreover, we assume that v ranges in all of $[1,\infty)$. On the ingoing null segment $v = 1$, we assume $\nu = -1$, ζ bounded, $r > c > 0$. This makes $\{v = 1\} \cap \mathcal{H} = (\tilde{U}, 1)$ for some $\tilde{U} < \infty$. We will assume $m \geq 0$ at $(\tilde{U}, 1)$.

We first show

[1] Compare with [1], where an additional inner boundary condition was imposed on a timelike curve "near" the event horizon. This was necessitated by the fact that the Regge-Wheeler coordinates used in [1] put the event horizon $\mathcal{H}$ at $r_* = -\infty$, even though $\mathcal{H}$ is part of the spacetime.

Proposition 1 *Under the above assumptions, $\nu < 0$ in $\mathcal{Q}$, and $\lambda > 0$ in $\mathcal{Q}\setminus\mathcal{H}$.*

Proof. Let $\mathcal{U}$ be the region

$$\mathcal{U} = \{p : \nu < 0 \text{ in } \overline{J^-(p)} \cap \mathcal{Q}\}$$

We first note that ν is continuous. For ν is assumed to be differentiable in the v direction, while differentiating (5) in the u direction one obtains an evolution equation for $\partial_u \nu$, which initially is 0. By continuity of ν, $\mathcal{U}$ is an open subset of $\mathcal{Q}$. In $\overline{\mathcal{U}} \cap \mathcal{Q}$, we clearly have $r \le R$, and $r < R$ in $\overline{\mathcal{U}} \cap \mathcal{Q} \setminus \Gamma$. Thus, it follows that $\lambda > 0$ in $\mathcal{U} \setminus \mathcal{H}$. For, if $\lambda(u', v') \le 0$, and $r(u', v') < R$, then $r(u', v^*) < R$ for all $v^* > v'$, and thus the curve $u = u'$ cannot intersect Γ.[2] Thus, by (7), we have $\partial_v \varpi \ge 0$ in $\overline{\mathcal{U}}$, and so, denoting $\Pi_1 = \inf_{u \in [1, \tilde{U}]} \varpi(u, 1)$ we have $\Pi_1 \le \varpi$ in $\overline{\mathcal{U}} \cap \mathcal{Q}$. Integrating $\partial_v r = \lambda$ from $v = 1$, it is clear that $r \ge c$. From (10), we have

$$\frac{\lambda}{1-\mu} \le \frac{1}{1 - \frac{M_0}{R}} \tag{12}$$

Integrating (5), it follows that

$$-\nu(u, v) \ge \tilde{C}(c, M_0, R, e, \Pi, v) > 0$$

on $\overline{\mathcal{U}} \cap \mathcal{Q}$. Thus $\mathcal{U} = \overline{\mathcal{U}} \cap \mathcal{Q} = \mathcal{Q}$. This completes the proof. $\square$

We note now that from the above Proposition and the results of [11], it follows that $\partial_v m \ge 0$, $\partial_u m \le 0$ in $\mathcal{Q}$, and thus, since $m(\tilde{U}, 1) \ge 0$, we have $m \ge 0$ throughout $\mathcal{Q}$. This immediately yields

$$0 \le 1 - \mu \le 1, \tag{13}$$

where we use the fact that λ and $1 - \mu$ have the same sign to obtain the lower bound.

3 The decay theorem

We will show the following

Theorem 1 *For any $p > 0$, there exists a constant $C(p)$ such that*

$$|\theta(u, v)| \le C v^{-p}.$$

Consider the statement

(S_q) There exists a constant $C(q)$ such that

$$|\theta(u, v)| \le C v^{-q}$$

$$\int_v^\infty \theta^2(\tilde{U}, \bar{v}) d\bar{v} \le C v^{-2q}. \tag{14}$$

[2] This statement follows from the Raychaudhuri equation $\partial_v(\Omega^{-2}\partial_v r) = -r\Omega^{-2} T_{vv} \le 0$. See [11]

We shall prove the following two propositions:

Proposition 2 *S_0 is true.*

Proposition 3 *S_q implies $S_{q+\frac{1}{4}}$.*

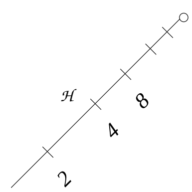
The above two propositions and the Archimedean property of the real numbers immediately yield the Theorem. □

Proof of Proposition 2. We note that Proposition 1 gives all the information necessary to apply Section 5 of [10]. The propositions of that section immediately yield the result of this proposition. In the process, one obtains a lower bound

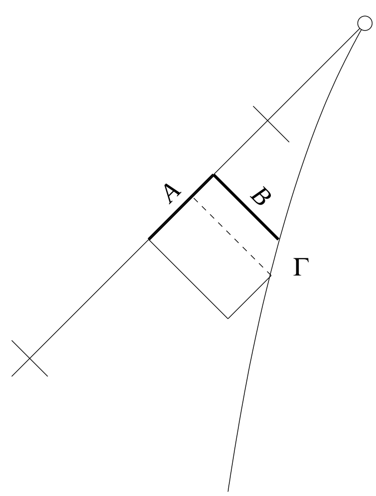
$$\frac{\lambda}{1-\mu} \geq c' > 0, \tag{15}$$

which we will require below. □

Proof of Proposition 3. Consider a dyadic decomposition of the event horizon, with respect to the v coordinate, as in [10].



In each dyadic interval, we can select, again as in [10], a subinterval A, split into two pieces as shown below, each of length $\sim v^{\frac{1}{2}}$,



such that

$$\int_A \theta^2 \leq C_1 v^{-\left(2q+\frac{1}{2}\right)}.$$

Here v denotes the v-value of any point in the dyadic interval. (Recall that these values are all comparable to the length of the dyadic interval.) We note that the assumptions of Propositions of Section 6 of [10] hold. We thus obtain the bound

$$\left|\frac{\zeta}{\nu}\right| \leq C_2 v^{-\left(q+\frac{1}{4}\right)}$$

on the interval B. It follows immediately, in view of the bounds $c \leq r \leq R$, and (13) that,

$$\int_B \left(\frac{\zeta}{\nu}\right)^2 (1-\mu)(-\nu)du \leq C_3 v^{-\left(2q+\frac{1}{2}\right)}.$$

Now, by the inequalities $\partial_v \varpi \geq 0$, $\partial_u \varpi \leq 0$, the fact that m is nonincreasing to the future along Γ, and the equation (7), this implies immediately

$$\int_v^\infty \frac{1-\mu}{\lambda}\theta^2(\tilde{U},\bar{v})d\bar{v} \leq C_4 v^{-\left(2q+\frac{1}{2}\right)} \tag{16}$$

along the event horizon. By the dyadic nature of the decomposition, (16) holds for all values of v, and not just for the v-values of the future endpoints of A. Finally, by (12), this proves immediately (14). Applying again the results of Section 6 of [10], now with *any* subinterval of the event horizon of length $\sim v$, in view of (14), we obtain that the inequality

$$\left|\frac{\zeta}{\nu}\right| \leq C_5 v^{-\left(q+\frac{1}{4}\right)} \tag{17}$$

holds throughout $\mathcal{Q}$, in particular, on Γ. The boundary conditions, namely $\lambda = 1$, $R = C$, m is non-increasing on Γ, yield that

$$|\theta(u,v)| \leq \left|\frac{\zeta}{\nu}\right|$$

on Γ, and thus by (17) we have

$$|\theta| \leq C_5 v^{-\left(q+\frac{1}{4}\right)}$$

on Γ. Integrating now the equation

$$\partial_u\theta = -\frac{\zeta}{\nu}\frac{\lambda}{1-\mu}(1-\mu)r\nu$$

from Γ, in view of the bounds (13), (12), and (17), we obtain

$$|\theta| \leq C_6 v^{-\left(q+\frac{1}{4}\right)}$$

throughout $\mathcal{Q}$. This concludes the proof. $\square$

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